ON THE STABILITY OF A HEAVY SYMMETRICAL GYROSCOPE ON GIMBALS

(OB USTOICHIVOSTI DVIZHENIIA TIAZHELOGO SIMMETRICHNOGO GIROSKOPA V KARDANOVOM PODVESE)

PMM Vol.22, No.2, 1958, pp.173-178

K. MAGNUS
(Freiburg, G.F.R.)

(Received 14 December 1957)

Stability conditions for the vertical position of a heavy symmetrical gyroscope (Lagrange's case) were derived by Chetaev [1], who used the second method of Liapunov without any simplifying assumptions (as it is used for example in the small vibrations method), and also by Rumiantsev [2], who found stability conditions for Kovalevska's case. This note supplements the above-mentioned results, presenting the derivation of stability conditions for the vertical position of a heavy symmetrical gyroscope on gimbals [in a Cardan mounting]. The solution of this problem can be regarded as a generalization of the Lagrange case, and it shows the existence of certain new effects noticed previously by Nikolai [3], when he investigated a special case of a rapidly rotating astatic gyroscope.

1. Statement of the problem. Assume that the gyroscope is suspended on gimbals (see Fig.1) in such a way that the outer axis of the gimbal system is vertical, i.e. it coincides with the direction of the gravitational force. The friction in the gimbal bearings is neglected. The system consists of a rotor, and the outer and inner gimbal rings, and it has three degrees of freedom. Its position can be uniquely determined by three coordinates. We shall use Euler's coordinate system consisting of the angles ϕ, ψ and θ, the initial position being when the planes of both rings coincide. We assume, as in the Lagrange case, that the center of gravity of both the rotor and the inner gimbal ring is on the axis of the rotor at a distance s from the stationary point of the suspension. We shall regard s as positive if in the zero position of the system the center of gravity is above the stationary point. The position of the center of gravity of the outer gimbal ring does not require any special assumptions. In the zero position $\psi = \theta = 0$ and the moments of inertia about the axes x, y, z, as shown in Fig.1, are the principal moments of inertia and are denoted as follows:

Reprint Order No. PMM 19.

A_1, $B_1 = A_1$,	C_1	for the rotor
A_2, B_2	C_2	for the inner gimbal ring
- -	C_3	for the outer gimbal ring

2. The initial integrals. Lagrange's problem of the motion of a heavy symmetrical gyroscope can be solved by quadratures because it is possible to find the first three integrals of the equations of motion. These integrals have a physical meaning. They express:

(1) That the vertical component of the angular momentum is constant

(2) That the component of the angular momentum in the direction of the gyroscope's axis is constant

(3) That the total energy of the whole system is constant.

These three integrals could also be found in the case of a heavy symmetrical gyroscope on gimbals.

The system is conservative because the moments arising from friction in the bearings of the gimbal system are neglected.

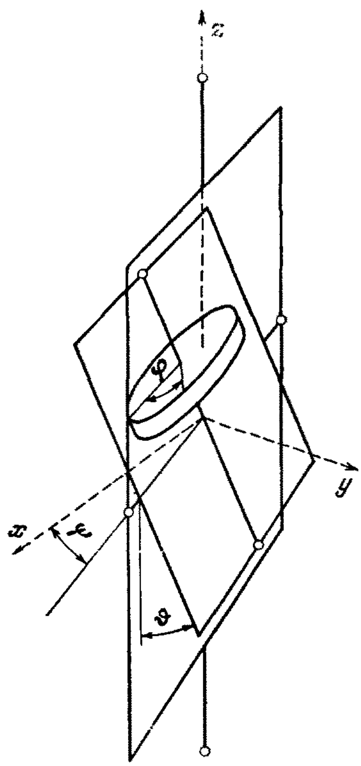

Fig. 1.

It can be shown easily that for a gyroscope on gimbals the vertical component of the angular momentum must also be constant. The time derivative of the vector of angular momentum equals the sum of the vectors of the moments of outside forces. The moment of the gravitational forces and the moment transmitted by the outer axis of the gimbals are acting on the system. Vectors of these moments must lie in a horizontal plane; therefore the tip of the vector of angular momentum can move only in a horizontal plane, and hence the vertical component of the total angular momentum must be a constant.

The third integral shows that the component of the rotor's angular momentum along its axis is constant (the total angular momentum not being constant). The vector of the moment of gravity forces about the rotor's axis is directed along the inner axis of the gimbal system (the so-called nodal line), and the vector of the moment transmitted from the rotor to the inner gimbal ring is perpendicular to the rotor's axis because of the absence of frictional forces. In the coordinate system attached to the rotor the relationship between the angular momentum and outside moments is as follows:

$$\frac{d\overline{D}_1}{dt} + [\overline{\omega}_1 \overline{D}_1] = \Sigma \overline{M}$$

($\overline{D}_1$ is the vector of the rotor's angular momentum and $\overline{\omega}_1$ is the vector of the rotor's angular velocity). On account of the symmetry of the rotor, the component of the Coriolis term $[\overline{\omega}_1 \overline{D}_1]$ along the rotor's axis vanishes as well as both moments. The tip of the vector of the angular momentum $\overline{D}_1$ moves in a plane normal to the rotor's axis; hence its component along the rotor's axis must be constant.

In our case the three integrals are as follows

$$\dot{\varphi} + \dot{\psi} \cos\vartheta = r_0 \tag{2.1}$$

$$\dot{\psi}\,[(A_1 + B_2 + C_3) - (A_1 + B_2 - C_2)\cos^2\vartheta] + C_1 r_0 \cos\vartheta = D_z \tag{2.2}$$

$$\begin{aligned}\dot{\vartheta}^2\,(A_1 + A_2) + \dot{\psi}^2[(A_1 + B_2 + C_3) - (A_1 + B_2 - C_2)\cos^2\vartheta] + \\ + C_1 r_0^2 + 2\,mgs\cos\vartheta = E\end{aligned} \tag{2.3}$$

where r_0 is the component of the rotor's angluar velocity along its axis, D_z is the vertical component of the total angular momentum and E is the double sum of kinetic and potential energies. The three integrals (2.1), (2.2), (2.3) reduce to the integrals of the Lagrange case when the masses of the gimbal rings vanish ($A_2 = B_2 = C_2 = C_3 = 0$). It can be shown that the problem of motion of a gyroscope on gimbals leads to quadratures as in the Lagrange case which we shall not present. Here we are interested only in the stability problem.

Let us introduce a new variable

$$u = \cos\vartheta \tag{2.4}$$

and also the following substitutions:

$$\frac{C_1 r_0}{A_1 + A_2} = k_1, \qquad \frac{2mgs}{A_1 + A_2} = k_3, \qquad \frac{A_1 + B_2 + C_3}{A_1 + A_2} = k_5$$

$$\frac{D_z}{A_1 + A_2} = k_2, \qquad \frac{E - C_1 r_0^2}{A_1 + A_2} = k_4, \qquad \frac{A_1 + B_2 - C_2}{A_1 + A_2} = k_6 \tag{2.5}$$

Then the original integrals reduce to:

$$\dot\varphi + \dot\psi u = r_0 \tag{2.6}$$

$$\dot\psi\,(k_5 - k_6 u^2) + k_1 u = k_2 \tag{2.7}$$

$$\dot\vartheta^2 + \dot\psi^2 (k_5 - k_6 u^2) + k_3 u = k_4 \tag{2.8}$$

We are interested in the stability of the system when the rotor's axis is vertical, a case represented by the conditions

$$\vartheta = 0 \quad (u = 1), \qquad \dot\vartheta = 0, \qquad \dot\varphi = \dot\varphi_0, \qquad \dot\psi = \dot\psi_0 \tag{2.9}$$

The above conditions define in equations (2.6), (2.7), (2.8) the constants r_0, k_2 and k_4 which depend on initial conditions. The constant k_1 also depends on initial conditions but it is already defined through r_0.

In order to investigate stability we must rewrite the integrals (2.6), (2.7) and (2.8) and introduce in them perturbations in the form of variations x_i in the variables

$$\dot\vartheta = 0 + x_1, \qquad \dot\varphi = \dot\varphi_0 + x_2, \qquad \dot\psi = \dot\psi_0 + x_3, \qquad u = 1 - x_4 \tag{2.10}$$

The initial conditions of a perturbed motion are changes; hence the constants may also assume different values:

$$r_0 = r_{00} + R, \quad k_1 = k_{10} + K_1, \quad k_2 = k_{20} + K_2, \quad k_4 = k_{40} + K_4$$

For a perturbed motion we obtain the following integrals:

$$x_2 + x_3 - \dot\psi_0 x_4 - x_3 x_4 = R \tag{2.11}$$

$$\begin{aligned} &- x_3 x_4^2 k_6 + x_3 x_4 2k_6 - x_4^2 \dot\psi_0 k_6 + x_3 (k_5 - k_6) + \\ &+ x_4 (2\dot\psi_0 k_6 - k_1) = K_2 - K_1 \end{aligned} \tag{2.12}$$

$$\begin{aligned} &x_1^2 - x_3^2 x_4^2 k_6 + x_3^2 x_4 2k_6 - x_3 x_4^2 2\dot\psi_0 k_6 + x_3^2 (k_5 - k_6) + x_3 x_4 4\dot\psi k_6 - \\ &- x_4^2 \dot\psi^2{}_0 k_6 + x_3 2\dot\psi_0 (k_5 - k_6) + x_4 (2\dot\psi_0^2 k_6 - k_3) - K_4 \end{aligned} \tag{2.13}$$

3. Construction of the Liapunov function. We shall seek the Liapunov function V of the variables x_i in the form of a linear combination of the first integrals in (2.11), (2.12), (2.13) respectively. It is convenient to introduce a new variable x_5, defined as follows:

$$x_5^2 = 2x_4 - x_4^2 = 1 - u^2 > 0 \tag{3.1}$$

By introducing a new constant K_0, we can rewrite (3.1) as follows:

$$x_4^2 + x_5^2 - 2x_4 = K_0 = 0 \tag{3.2}$$

We can now construct the Liapunov function V in the form

$$V = K_4 + \alpha_1 (K_2 - K_1) + \alpha_2 K_0 + \alpha_3 R^2 \tag{3.3}$$

The constants α_1, α_2, α_3 can be regarded for the time being as arbitrary. The time derivative of V vanishes identically because every term of (3.4) is a constant. By Liapunov's theorem, the conditions for which V is positive definite, determine the sufficient conditions for the stability of the investigated system.

Substituting (2.11), (2.12), (2.13) and (3.2) in (3.3) we obtain

$$\begin{aligned} V = x_1^2 + x_2^2\alpha_3 + x_3^2(\alpha_3 + k_5 - k_6) + x_4^2(\alpha_2 + \alpha_3\dot\psi_0^2) + \\ + x_5^2(\dot\psi_0^2 k_6 + \alpha_1\dot\psi_0 k_6 + \alpha_2) + x_2x_3 2\alpha_3 - x_2x_4 2\dot\psi_0\alpha_3 - \\ - x_3x_4 2\alpha_3\dot\psi_0 + x_3(2\dot\psi_0 + \alpha_1)(k_5 - k_6) - x_4(k_3 + \alpha_1 k_1 + 2\alpha_2) + \\ + x_3x_4^2 2\dot\psi_0\alpha_3 - x_3^2x_4 2\alpha_3 + x_3x_5^2(2\dot\psi_0 k_6 + \alpha_1 k_6) - \\ - x_2x_3x_4 2\alpha_3 + x_3^2x_4^2\alpha_3 + x_3^2x_5^2 k_6 \end{aligned} \tag{3.4}$$

The linear terms of this expression vanish when

$$\alpha_1 = -2\dot\psi_0, \qquad \alpha_2 = \dot\psi_0 k_1 - \tfrac{1}{2}k_3 \tag{3.5}$$

In such a case the Liapunov function (3.4) becomes

$$\begin{aligned} V = x_1^2 + (x_2 + x_3)^2\alpha_3 + x_3^2(k_5 - k_6) + x_4^2(\dot\psi_0 k_1 - \tfrac{1}{2}k_3 + \alpha_3\dot\psi_0^2) + \\ + x_5^2(\dot\psi_0 k_1 - \tfrac{1}{2}k_3 - \dot\psi_0^2 k_6 - (x_2 + x_3)x_4 2\dot\psi_0\alpha_3 + \\ + [x_3x_4^2 2\dot\psi_0\alpha_3 - x_3^2x_4 2\alpha_3 - x_2x_3x_4 2\alpha_3 + x_3^2x_4^2\alpha_3 + x_3^2x_5^2 k_6] \end{aligned} \tag{3.6}$$

The Liapunov function V consists then of a quadratic form of the variables x_i plus the third- and the fourth-order terms.

The quadratic form is positive definite if the following inequalities are satisfied:

$$\alpha_3 > 0, \quad k_5 - k_6 > 0, \quad \dot\psi_0 k_1 - \tfrac{1}{2}k_3 > 0, \quad \dot\psi_0 k_1 - \tfrac{1}{2}k_3 - \dot\psi_0^2 k_6 > 0 \tag{3.7}$$

When the above conditions are satisfied, the multipliers of the pure quadratic terms are positive and in addition the Sylvester's inequality for the quadratic form of the variables $x_2 + x_3$ and x_4 is satisfied:

$$\begin{vmatrix} \alpha_3 & \dot{\psi}_0\alpha_3 \\ \dot{\psi}_0\alpha_3 & (\dot{\psi}_0 k_1 - \frac{1}{2} k_3 + \alpha_3\dot{\psi}_0^2) \end{vmatrix} = \alpha_3(\dot{\psi}_0 k_1 - \frac{1}{2} k_3) > 0 \tag{3.8}$$

The first condition (3.7) could be always satisfied by choosing a suitable value for the constant α_3; the second condition (3.7) is automatically satisfied because of (2.5), as $k_6 > 0$; the third condition (3.7) is obviously satisfied if the fourth condition (3.7) is satisfied. Thus, the conditions (3.7) turn out to be sufficient for the stability of a gyroscope with respect to the variables θ, $\dot{\theta}$, $\dot{\phi}$, $\dot{\psi}$.

4. Conditions of stability. The inequality (3.8) can be considered as the relation determining the admissible values of the quantity $\dot{\psi}_0$. Boundaries of the admissible region are as follows:

$$\left.\begin{matrix} \dot{\psi}_{01} \\ \dot{\psi}_{02} \end{matrix}\right\} = \frac{k_1}{2k_6}\left[1 \pm \sqrt{1 - \frac{2k_3k_6}{k_1^2}}\right] \tag{4.1}$$

or, using the original symbols

$$\left.\begin{matrix} \dot{\psi}_{01} \\ \dot{\psi}_{02} \end{matrix}\right\} = \frac{C_1 r_0}{2(A_1 + B_2 - C_3)}\left[1 \pm \sqrt{1 - \frac{4mgs\,(A_1 + B_2 - C_2)}{C_1^2 r_0^2}}\right] \tag{4.2}$$

The solutions are real when

$$C_1^2 r_0^2 > 4mgs\,(A_1 + B_2 - C_2) \tag{4.3}$$

and the real solutions correspond to a real physical motion.

When $B_2 = C_2 = 0$, then the expression (4.3) reduces to the known expression representing the necessary and sufficient conditions for stability in the Lagrange case. For a gyroscope on gimbals this condition is necessary but not sufficient. In order to have stability, one more condition must be satisfied, namely

$$\dot{\psi}_{02} \leqslant \dot{\psi}_0 \leqslant \dot{\psi}_{01} \tag{4.4}$$

When the rotor is in vertical position, the rotational velocity of the gimbal system turns out to be the deciding factor, in the problem of stability of a gyroscope on gimbals. This result is physically plausible if we take into account that in the Lagrange case the quantities ψ and $\dot{\psi}$ are undefined when the rotor is in vertical position, and the nodal line and its azimuthal rotational velocity are only fixed parameters, whereas for a gyroscope on gimbals the angle ψ becomes an important variable because it defines the motion of the gimbal system.

Assuming, without any loss of generality, that $r_0 > 0$, we obtain for a "standing gyroscope" ($s > 0$)

$$0 < \dot{\psi}_{02} < \dot{\psi}_{01}$$

It is seen that a standing gyroscope on gimbals may lose stability without an initial push on the gimbal ring in the direction of rotor's spin. For a "hanging gyroscope" ($s < 0$) we have:

$$\dot{\psi}_{02} < 0 < \dot{\psi}_{01}$$

In the last case with $\dot{\psi}_0 = 0$ the motion is stable. In the Lagrange case the hanging gyroscope is always stable, but a hanging gyroscope on gimbals could lose stability when it receives a push greater than $\dot{\psi}_{02}$ in the direction opposite to its spin, or a push greater than $\dot{\psi}_{01}$ in the direction of its spin. The proof of instability in these two cases will not be here presented.

The influence of the quantity $\dot{\psi}_0$ on the stability could be easily demonstrated on appropriate models. One more remark should be added; namely, that the limiting values (4.2) for the azimuthal velocities correspond to velocities of "regular precession" of a heavy gyroscope on gimbals as $\theta \to 0$.

BIBLIOGRAPHY

1. Chetaev, N.G., Ob ustoichivosti vrashchenia tverdogo tela s odnoi nepodvizhnoi tochkoi v sluchae Lagranzha (On the stability of revolution of a rigid body with one fixed point in the Lagrange case). *PMM* Vol. 18, No. 1, pp. 123-124, 1954.

2. Rumiantsev, V.V., Ob ustoichivosti vrashchenia tverdogo tela s odnoi nepodvizhnoi tochkoi v sluchae S.V. Kovalevskoi (On the stability of revolution of a rigid body with one fixed point in Kovalevska's case). *PMM* Vol. 18, No. 6, pp. 457-458, 1954.

3. Nikolai, E.L., O dvizhenii uravnoveshennogo giroskopa v kardanovom podvese (On the motion of a balanced gyroscope on gimbals). *PMM* Vol. 3, No. 4, pp. 3-34, 1939.

Translated by T.L.